GRAPHS WHOSE CIRCULAR CHROMATIC NUMBER EQUALS THE CHROMATIC NUMBER

XUDING ZHU*

This paper presents a method for constructing graphs whose circular chromatic number equals their chromatic number.

1. Introduction

The circular chromatic number of a graph is a natural generalization of the chromatic number of a graph introduced by Vince in 1988 [13], under the name "the star chromatic number". Given two integers k, d such that $k \geq d$, a (k, d)-coloring of a graph G is a coloring ϕ of the vertices of G with colors $0, 1, 2, \cdots, k-1$ such that for any two adjacent vertices x and y of G, we have $d \leq |\phi(x) - \phi(y)| \leq k - d$. The *circular chromatic number* $\chi_c(G)$ of G is defined as the infimum of the ratio k/d for which there exists a (k, d)-coloring of G. It was shown by Vince [13] that for finite graphs, the infimum in this definition is always attained, and hence it can be replaced by minimum.

Note that a $(k, 1)$-coloring of a graph G is just an ordinary k-coloring of G. This implies that $\chi_c(G) \leq \chi(G)$ for any graph G. On the other hand, it was shown in [13] that $\chi_c(G) > \chi(G) - 1$. Therefore $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. Hence $\chi(G) = \lceil \chi_c(G) \rceil$ for any graph G. In this sense, the circular chromatic number of a graph is a refinement of its chromatic number, and the chromatic number can be regarded as an approximation of its circular chromatic number. The concept of circular chromatic number of a graph has been studied in many articles [1-2,3-4,7-10,]. The focus has been on the relation between the chromatic number of a graph and its circular chromatic number, the existence of graphs of special properties with given

circular chromatic number, and the circular chromatic number of special classes of graphs. For a survey on the subject of circular chromatic number, see [21].

It is well known [2, 13] that there are graphs G such that $\chi_c(G) = \chi(G)$, and there are also graphs G such that $\chi_c(G)$ is arbitrarily close to $\chi(G) - 1$. In this sense the above upper and lower bound for $\chi_c(G)$ in terms of $\chi(G)$ are sharp. The question regarding which graphs G satisfy the equality $\chi_c(G) = \chi(G)$ was asked by Vince [13] and has been studied in [1, 5, 12, 14, 15, 16]. It was proved by Guichard that it is NP-hard to decide whether or not an arbitrary graph G satisfies $\chi_c(G) = \chi(G)$. There are some known sufficient conditions under which $\chi_c(G) = \chi(G)$ [1, 5, 12, 14]. However, these sufficient conditions are very restrictive. This paper presents a method for constructing graphs G such that $\chi_c(G) = \chi(G)$. This method enables us to obtain many new graphs G with $\chi_c(G) = \chi(G)$. In particular, we shall use this method to construct planar graphs G with $\chi_c(G) = \chi(G)$, which answers some questions raised in [12].

We shall use the following equivalent definition of the circular chromatic number of a graph given in [14]:

Suppose $r \geq 1$ is a real number. Let C^r be a circle of length r. An r-circular coloring of a graph G is a mapping ϕ which maps each vertex x of G to a unit length open arc $\phi(x)$ of C^r such that $\phi(x) \cap \phi(y) = \emptyset$ whenever xy is an edge of G. The circular chromatic number of G is the infimum of those r for which there exists an r-circular coloring of G.

2. The construction method

This section presents a method of constructing new graphs G satisfying $\chi_c(G) = \chi(G)$ from old ones.

Definition 1. Suppose G is a graph and x, y are non-adjacent vertices of G. We say the triple $(G; x, y)$ is an n-circular superedge if $G + xy$ is n-colorable and that for any $\epsilon > 0$ and for any $(n - \epsilon)$-circular coloring ϕ of G, we have $\phi(x) \cap \phi(y) = \emptyset$.

Suppose H is a graph and ab is an edge of H. We shall denote by $H[ab, (G; x, y)]$ the graph obtained from H by replacing the edge ab with the superedge $(G; x, y)$, i.e., $H[ab, (G; x, y)]$ is obtained from the disjoint union of H and G, by deleting the edge ab, identifying a with x, and identifying b with y.

Theorem 1. *Suppose $(G; x, y)$ is an n-circular superedge. Then for any graph H with $\chi_c(H) = \chi(H) = n$, and for any edge ab of H, we have $\chi_c(H[ab, (G; x, y)]) = \chi(H[ab, (G; x, y)]) = n$.*

Proof. It is easy to see that $\chi(H[ab, (G; x, y)]) \leq n$. Indeed, G has an n-coloring f such that $f(x) \neq f(y)$ (because $G + xy$ is n-colorable) and H has an n-coloring g such that $g(a) \neq g(b)$. We may assume that f and g use the same n colors and that $f(x) = g(a)$, $f(y) = g(b)$. Then the union of f and g is an n-coloring of $H[ab, (G; x, y)]$.

Now we shall prove that $\chi_c(H[ab,(G;x,y)]) \geq n$, and hence $\chi_c(H[ab,(G;x,y)]) = \chi(H[ab,(G;x,y)]) = n$. Assume to the contrary that $\chi_c(H[ab,(G;x,y)]) < n$. Then there is an $\epsilon > 0$ such that $\chi_c(H[ab,(G;x,y)]) \leq n - \epsilon$. Let ϕ be an $(n-\epsilon)$-circular coloring of $H[ab,(G;x,y)]$. Then the restriction of ϕ to G is an $(n-\epsilon)$-circular coloring of G. Hence $\phi(x) \cap \phi(y) = \emptyset$. This means that the restriction of ϕ to the vertices of H is indeed an $(n-\epsilon)$-circular coloring of H, contrary to the assumption that $\chi_c(H) = n$. ∎

Once we have one graph H of circular chromatic number n and one n-circular superedge $(G;x,y)$, we may construct infinitely many new graphs G' with $\chi_c(G') = n$, by repeatedly applying Theorem 1. It is easy to find an H with $\chi_c(H) = n$, for example, we may take $H = K_n$. The next section will discuss the construction of n-circular superedges. Before doing that, we make a comparison of the construction method presented in this section with the Hajós' sum, which is used to construct new n-chromatic graphs from existing ones.

Suppose G_1 and G_2 are graphs and $e = ab$ is an edge of G_1, $e' = cd$ is an edge of G_2. The *Hajós' sum* of G_1 and G_2 is the graph obtained from the disjoint union of G_1 and G_2 by identifying a and c, deleting the edges e and e', and adding the edge bd. Hajós [6] proved that all graphs G of chromatic number at least n can be recursively constructed from copies of K_n by three operations: adding vertices and edges; identifying non-adjacent vertices; and taking the Hajós' sum.

The operations of adding vertices and edges, identifying non-adjacent vertices do not decrease the circular chromatic number of a graph. However, the operation of taking the Hajós' sum may decrease the circular chromatic number, although it does not decrease the chromatic number. If we attempt to find a method of recursively constructing all graphs of circular chromatic number at least n, as an analogue for the circular chromatic number of Hajós' Theorem, we need to replace the Hajós' sum by some other operations. For this purpose, we define (or interpret) the Hajós' sum in a different way.

Suppose G is a graph and x,y are non-adjacent vertices of G. We say the triple $(G;x,y)$ is an *n-superedge* if for any $(n-1)$-coloring ϕ of G, we have $\phi(x) \neq \phi(y)$. In particular, if $\chi(G) \geq n$, then for any pair x,y of non-adjacent vertices of G, $(G;x,y)$ is an n-superedge. It is easy to see that a triple $(G;x,y)$ is an n-superedge if and only if $\chi(G|_{xy}) \geq n$, where $G|_{xy}$ is the graph obtained from G by identifying the two vertices x and y into a single vertex.

Suppose G is a graph of chromatic number at least n, and xx' is an edge of G. Let G^* be the graph obtained from G by deleting the edge xx', adding a new vertex y and an edge $x'y$. Then $G^*|_{xy} = G$, therefore $(G^*;x,y)$ is an n-superedge. We call the triple $(G^*;x,y)$ a *special n-superedge* associated with G.

By this definition, taking the Hajós' sum of two graphs of chromatic number at least n is the same as replacing an edge of one graph by a special n-superedge associated with the other. Therefore Hajós' Theorem can be stated as follows:

Theorem 2. *Let $\mathcal{G}_n$ be the minimal class of graphs which contains K_n and is closed under the following operations:*

1. *adding vertices and edges;*
2. *identifying non-adjacent vertices;*
3. *replacing an edge by a special n-superedge associated with a member of $\mathcal{G}_n$.*

Then $\mathcal{G}_n$ is exactly the class of graphs with chromatic number at least n.

With this interpretation of Hajós' Theorem, to find an analogous statement for the circular chromatic number probably amounts to find a class of special n-circular superedges which can be recursively constructed from K_n. The next section discusses the construction of n-circular superedges. We shall present a class of special n-circular superedges which are recursively constructed from K_n. However, this class is probably not large enough for the construction of all graphs of circular chromatic number at least n. Neverthless, we do construct many interesting examples by using this class of special n-circular superedges.

3. n-circular superedges

First we give a result which shows that there are plenty of n-circular superedges $(G; x, y)$.

Theorem 3. *Suppose G is an n-chromatic graph, and that x, y are two non-adjacent vertices of G. If $\chi(G|_{xy}) = n+1$, then $(G; x, y)$ is an n-circular superedge.*

Proof. Let f be an n-coloring of G. Then we must have $f(x) \neq f(y)$, for otherwise f is an n-coloring of $G|_{xy}$. Therefore $G + xy$ is n-colorable.

Assume to the contrary that $(G; x, y)$ is not an n-circular superedge. Then there exists an $\epsilon > 0$ and an $(n-\epsilon)$-circular coloring ϕ of G such that $\phi(x) \cap \phi(y) \neq \emptyset$. Let $p_0 \in \phi(x) \cap \phi(y)$. Starting from p_0, we put n points $p_0, p_1, \cdots, p_{n-1}$ on the circle $C^{n-\epsilon}$, such that the arc $[p_i, p_{i+1}]$ (the arc from p_i to p_{i+1} along the clockwise direction) has length $(n-\epsilon)/n < 1$. Then for each vertex v of G, $\phi(v)$ contains at least one of the points p_i, because $\phi(v)$ is a unit length arc. Now we define an n-coloring f of G, by letting $f(v) = \min\{i : p_i \in \phi(v)\}$. Then f is an n-coloring of G for which $f(x) = f(y)$, contrary to the assumption that $\chi(G|_{xy}) = n+1$. ∎

Applying Theorem 3, one can easily find an n-circular superedge. For example, suppose G is a uniquely n-colorable graph which is not a complete n-partite graph. Then there are two vertices x and y of G which are in different color classes of G (of an n-coloring of G) and which are non-adjacent. Then the triple $(G; x, y)$ is an n-circular superedge. The condition of Theorem 3 is sufficient for a triple to be an n-circular superedge. However, it is not a necessary condition, as shown by the results below. Another disadvantage of the condition of Theorem 3 is that these graphs are not easily constructed. Our next result gives a method of recursively constructing an infinite family of n-circular superedges, starting from K_n.

Definition 2. Suppose G is a graph and x, x' are non-adjacent vertices of G. We say x' is a *partner* (with respect to integer n) of x if $\chi_c(G + xx') = n$. The set of partners of x is denoted by $P(x)$.

Theorem 4. *Suppose G is a graph, and x, y are two non-adjacent vertices of G, and that $G + xy$ is n-colorable. If the subgraph of G induced by the set $P(x) \cup \{y\}$ is not bipartite, then $(G; x, y)$ is an n-circular superedge.*

Proof. It suffices to show that for any $\epsilon > 0$ and for any $(n - \epsilon)$-circular coloring ϕ of G, we have $\phi(x) \cap \phi(y) = \emptyset$. Assume to the contrary that there exists a $(n - \epsilon)$-circular coloring ϕ of G such that $\phi(x) \cap \phi(y) \neq \emptyset$. For each $x' \in P(x)$, we must have $\phi(x) \cap \phi(x') \neq \emptyset$, for otherwise ϕ would be a $(n - \epsilon)$-circular coloring of $G + xx'$, contrary to the definition of partner. Let p, q be the two end points of $\phi(x)$. First we note that for any non-isolated vertex $v \in P(x) \cup \{y\}$, $\phi(v)$ contains exactly one of the points p, q. Indeed, because $\phi(v)$ intersects $\phi(x)$ and both $\phi(v)$ and $\phi(x)$ are unit length arcs, it follows that either $\phi(v) = \phi(x)$, or $\phi(v)$ contains exactly one of the points p, q. However, as v has a neighbour, say u, in $P(x) \cup \{y\}$, which implies that $\phi(u) \cap \phi(v) = \emptyset$ and $\phi(u) \cap \phi(x) \neq \emptyset$, we conclude that $\phi(v) \neq \phi(x)$.

For two adjacent vertices u, v of $P(x) \cup \{y\}$, the two arcs $\phi(u), \phi(v)$ are disjoint, hence one of them contains p, the other contains q. We may think of p, q as colors assigned to the non-isolated vertices of $P(x) \cup \{y\}$, which is then a proper 2-coloring of the subgraph of G induced by the set of non-isolated vertices of $P(x) \cup \{y\}$, contrary to our assumption. Therefore the circular coloring ϕ does not exist, and hence $(G; x, y)$ is an n-circular superedge. ∎

Theorem 4 can be easily used to construct n-circular superedges. Take any graph G of circular chromatic number n, say let $G = K_n$. Let xx' be an edge of G, and let $G - xx'$ be the graph obtained from G by deleting the edge xx'. Then x' is a partner of x in $G - xx'$. Take an odd cycle, say $C_{2k+1} = (c_0, c_1, \cdots, c_{2k})$. Take $2k$ disjoint copies of $G - xx'$. Identify all x in these copies into a single vertex, name it x again. Identify the x' in the ith copy of $G - xx'$ with c_i. Let the resulting graph be H, and rename the vertex c_0 as y. Then $(H; x, y)$ is an n-circular superedge.

Fig. 1(a) below is a 4-circular superedge constructed by using two copies of $K_4 - xx'$ and a triangle. Fig. 1(b) is a 4-circular superedge constructed from 4 copies of K_4 and a C_5.
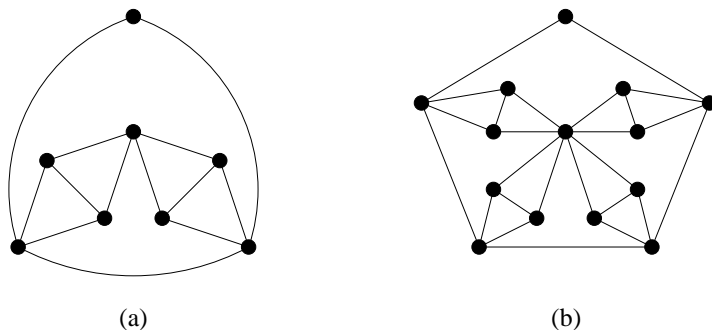

(a)   (b)

Fig. 1. Examples of circular superedges

It is easy to verify that the examples in Fig. 1 do not satisfy the condition of Theorem 3.

Theorem 5. *Suppose* $(G; x, y)$ *and* $(G'; x', y')$ *are both* n-*circular superedges. Then replace any edge* ab *of* G *with the superedge* $(G'; x', y')$, *the triple* $(G[ab, (G'; x', y')]; x, y)$ *is again an* n-*circular superedge.*

The proof of Theorem 5 is straightforward and omitted. Using Theorems 4 and 5, the construction of examples of n-circular superedges becomes very easy. It is natural that the n-circular superedges constructed by this method represent just a small portion of the family of n-circular superedges. The following are some examples of n-circular superedges that are not constructed by this method.

Theorem 6. *For* $k \geq 3$, *let* H_k *be the graph with* $2k+1$ *vertices,* $v_1, v_2, \cdots, v_{2k}, u$, *in which* $v_i v_j$ *is an edge if and only if* $|i - j| \leq k - 1$; *and* u *is adjacent to* $v_1, v_2, \cdots, v_{k-1}, v_{2k}$. *Let* $x = v_{2k}$ *and* $y = v_1$. *Then* $(H_k; x, y)$ *is a* $(k+1)$-*circular superedge.*

Proof. Fig. 2(a) is a depiction of H_k. It is easy to verify that $H_k + xy$ is $(k+1)$-colorable. It remains to show that for any $\epsilon > 0$ and for any $(k+1-\epsilon)$-circular coloring ϕ of H_k, we have $\phi(x) \cap \phi(y) = \emptyset$.

Assume to the contrary that ϕ is a $(k+1-\epsilon)$-circular coloring of H_k such that $\phi(x) \cap \phi(y) \neq \emptyset$. Assume that $\phi(x) = (p_x, q_x)$ and that $\phi(y) = (p_y, q_y)$, where we alway write an arc (p, q) in such a way that the direction of traversing the arc from p to q is the clockwise direction. Without loss of generality, we may assume that $p_y \in (p_x, q_x)$, hence $q_x \in (p_y, q_y)$. The partial coloring is shown in Fig. 2(b), where the arc named x is actually the arc $\phi(x)$, etc. Now the arc $\phi(v_{k+1})$ must have non-empty intersection with $\phi(x) = \phi(v_1)$, because otherwise $\phi(v_1), \phi(v_2), \cdots, \phi(v_{k+1})$ would be $k+1$ pairwise disjoint unit length arcs, contrary to the assumption that the circle has length $k+1-\epsilon$. Since v_{k+1} is adjacent to y, hence $\phi(v_{k+1}) \cap \phi(y) = \emptyset$, the position of $\phi(v_{k+1})$ must be as shown in Fig. 2(b).

Similarly, $\phi(v_k)$ has non-empty intersection with $\phi(y)$, and $\phi(v_k) \cap \phi(x) = \emptyset$. Hence the position of $\phi(v_k)$ is as shown in Fig. 2(b). Now $\phi(u)$ must have non-empty intersection with $\phi(v_k)$, for otherwise $\phi(u), \phi(v_1), \phi(v_2), \cdots, \phi(v_k)$ would be $k+1$ pairwise disjoint unit length arcs, contrary to the assumption that the circle has length $k+1-\epsilon$. As $\phi(u) \cap \phi(y) = \emptyset$, the position of $\phi(u)$ is as shown in Fig. 2(b). Suppose $\phi(v_{k+1}) = (p_{v_{k+1}}, q_{v_{k+1}})$ and $\phi(u) = (p_u, q_u)$. Let $p = p_{v_{k+1}}$ and $q = q_u$. The points p, q are shown in Fig. 2(b).

The arc (p, q) has length at least 3, because it contains three disjoint unit length arcs $\phi(v_{k+1}), \phi(y)$ and $\phi(u)$. Therefore the arc (q, p) has length less than $k - 2$. Consider the vertices $v_2, v_3, \cdots, v_{k-1}$. These vertices induce a clique of size $k - 2$. Hence the arcs $\phi(v_2), \phi(v_3), \cdots, \phi(v_{k-1})$ are pairwise disjoint. Each of the vertices $v_2, v_3, \cdots, v_{k-1}$ is adjacent to both v_{k+1} and u. Hence each of the arcs $\phi(v_2), \phi(v_3), \cdots, \phi(v_{k-1})$ is contained in (p, q) or (q, p). Since (q, p) has length less than $k - 2$, at least one of the arcs $\phi(v_2), \phi(v_3), \cdots, \phi(v_{k-1})$ is contained in (p, q).

However, it is easy to see from Fig. 2(b) that there is no room in (p, q) for any of these arcs, because each of the arcs $\phi(v_2), \phi(v_3), \cdots, \phi(v_{k-1})$ should be disjoint from $\phi(x)$ and $\phi(v_k)$. ∎

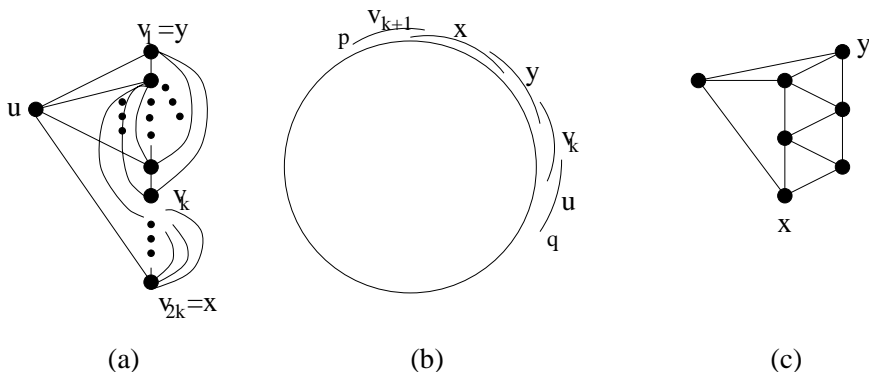Fig. 2(c) is the graph H_3, and $(H_3; x, y)$ is a 4-circular superedge.

Fig. 2. For the proof of Theorem 6

Theorem 7. *Suppose G is a graph with $\chi_c(G) = 3$, and that v is a vertex of G of degree 2. Let x, y be the two neighbours of v. If $G + xy$ is 3-colorable, then $(G - v; x, y)$ is a 3-circular superedge.*

Proof. It suffices to show that for any $\epsilon > 0$, and for any $(3 - \epsilon)$-circular coloring ϕ of $G - v$, we have $\phi(x) \cap \phi(y) = \emptyset$.

Assume to the contrary that ϕ is a $(3 - \epsilon)$-coloring of $G - v$ such that $\phi(x) \cap \phi(y) \neq \emptyset$. Then $\phi(x) \cap \phi(y)$ is a nonempty open interval, hence it has length $\delta > 0$. If $\delta \geq \epsilon$, then $C^{3-\epsilon} - (\phi(x) \cup \phi(y))$ is an arc of length at least 1. Let I be a unit length open arc contained in $C^{3-\epsilon} - (\phi(x) \cup \phi(y))$. We can define a $(3 - \epsilon)$-circular coloring ϕ' of G by letting $\phi'(v) = I$, and letting $\phi'(u) = \phi(u)$ for $u \neq v$, contrary to the assumption that $\chi_c(G) = 3$.

Assume that $\delta < \epsilon$. We cut the circle $C^{3-\epsilon}$ at a point $p \notin \phi(x) \cap \phi(y)$. Then insert an arc of length $\epsilon - \delta$ into the circle $C^{3-\epsilon}$ at p, obtaining a circle of length $3 - \delta$. Now we may think of ϕ as a mapping which assigns to each vertex u of G an arc $\phi(u)$ of $C^{3-\delta}$. If $\phi(u)$ does not contain the point p, then it still has unit length; if $\phi(u)$ does contain the point p, then the present length of $\phi(u)$ (i.e., the length of $\phi(u)$ after the insertion) is $1 + (\epsilon - \delta)$. Obviously we can remove a portion of $\phi(u)$ of length $\epsilon - \delta$ from one of its ends to obtain a unit length open arc, and hence obtain a $(3 - \delta)$-circular coloring ϕ' of $G - v$. Moreover, in this new circular coloring ϕ', we have $\phi'(x) \cap \phi'(y)$ is an interval of length δ. Using the previous argument, we obtain a $(3 - \delta)$-circular coloring of G, contrary to the assumption that $\chi_c(G) = 3$. ∎

The following theorem gives a necessary condition for a triple $(G; x, y)$ to be an n-circular superedge.

Theorem 8. *Suppose $(G; x, y)$ is an n-circular superedge. Then $\chi_c(G|_{xy}) \geq n$.*

Proof. If $\chi_c(G|_{xy}) < n$, then there exists an $\epsilon > 0$ and an $(n - \epsilon)$-circular coloring ϕ of $G|_{xy}$. Then ϕ induces an $(n - \epsilon)$-circular coloring of G in which $\phi(x) = \phi(y)$, contrary to the definition of n-circular superedges. ∎

However, this necessary condition is not sufficient. Let G be a path with 3 edges, and let x, y be the end vertices of the path. Then $\chi_c(G|_{xy}) = 3$, and it is easy to see that $(G; x, y)$ is not a 3-circular superedge. Indeed, for any $k \geq 3$, let Q_k be the graph obtained from the complete graph K_k by deleting an edge xy, adding a new vertex x' and an edge $x'y$. Then $Q_k|_{x'x}$ has circular chromatic number k, but $(Q_k; x', x)$ is not a k-circular superedge.

Among the n-circular superedges discussed above, the most promising one is the construction given in Theorem 4. We do not know whether or not, together with the operations of adding vertices and edges, and identifying non-adjacent vertices, these n-circular superedges are enough to construct all graphs G with $\chi_c(G) = n$. To be precise, we would like to ask Question 1 below.

First we observe that if, in the definition of n-circular superedge, we omit the requirement that $G + xy$ be n-colorable, then instead of constructing graphs H with $\chi_c(H) = n$, we would have constructed graphs H with $\chi_c(H) \geq n$. We shall be interested in the construction of graphs with circular chromatic number at least n in the following. For this purpose, we slightly change the definition of n-circular superedge, by omitting the requirement that $G + xy$ be n-colorable.

Given $2k$ graphs $G_1, G_2, \cdots, G_{2k}$. For $i = 1, 2, \cdots, 2k$, let $x_i y_i$ be an edge of G_i. Let $C_{2k+1} = (c_0, c_1, \cdots, c_{2k})$ be a cycle of length $2k+1$. Let H be the graph obtained from the disjoint union of $G_1, G_2, \cdots, G_{2k}, C_{2k+1}$, by identifying all the x_i's into a single vertex x, identifying y_i and c_i for $i = 1, 2, \cdots, 2k$. Rename the vertex c_0 with name y. If each of the graphs G_i has circular chromatic number at least n, we call the triple $(H; x, y)$ a special n-circular superedge associated with the graphs $G_1, G_2, \cdots, G_{2k}$.

Let $\mathcal{G}'_n$ be the smallest family of graphs which contains K_n, and is closed under the following operations:

1. adding vertices and edges;
2. identifying non-adjacent vertices;
3. replacing an edge by a special n-circular superedge associated with some members of $\mathcal{G}'_n$.

Then for $n \geq 3$, each graph $G \in \mathcal{G}'_n$ has circular chromatic number at least n. The question is:

Question 1. *Does $\mathcal{G}'_n$ contain all graphs G with $\chi_c(G) \geq n$? If $\mathcal{G}'_n$ does not contain all graphs G with $\chi_c(G) \geq n$, is it possible to find a larger family of special n-circular superedges so that all graphs with $\chi_c(G) \geq n$ can be constructed as above?*

A less ambitious objective is to find a method that constructs all graphs of circular chromatic number at least 3.

Concerning graphs with circular chromatic numbers at least 3, there are some interesting characterizations of such graphs which follow from a recent result of Brandt [3] and an earlier result of Pach [11]. Both Brandt and Pach were aiming at some other problems. However, once we start to look at their results from the point of view of circular colorings, the characterizations below follow immediately. We shall not cite their original results, but interpret them in the language of circular chromatic numbers.

Theorem 9. ([3]) *Let H be the graph obtained from the Petersen graph by deleting one vertex. A graph G has circular chromatic number at least 3 if and only if every maximal triangle free supergraph G' of G contains H as a subgraph.*

Theorem 10. ([11]) *A graph G has circular chromatic number at least 3 if and only if every maximal triangle free supergraph G' of G has an independent set whose elements do not have a common neighbour.*

Note that if G does contain a triangle, then the conditions of Theorems 9 and 10 are satisfied automatically, and such graphs have circular chromatic number at least 3.

These characterizations of graphs with circular chromatic number at least 3 seem very interesting. It might be helpful in the search for methods for constructing such graphs. It would also be interesting if these characterizations can be generalized to graphs with circular chromatic numbers at least n for arbitrary integer n.

Before closing this section, we point out that the proof of Theorem 4 can be easily adopted to prove the following theorem:

Theorem 11. *Suppose G is an n-chromatic graph. If there exists a vertex x of G such that the subgraph of G induced by $P(x)$ is not bipartite, then $\chi_c(G)=n$. Here $P(x)$ is the set of partners of x with respect to n.*

4. Planar graphs

In [12], the authors discussed the problem of the circular chromatic number of planar graphs. Let W_{2n+1} be the graph obtained from the odd cycle $C_{2n+1} = (x_0, x_1, \cdots, x_{2n})$, by adding a universal vertex u, i.e., add vertex u and edges ux_i $(i = 0, 1, \cdots, 2n)$. The graph W_{2n+1} is called the $(2n+1)$-*wheel*. The edges ux_i are called the *spokes* of W_{2n+1}. Let G_{2n+1} be the graph obtained from W_{2n+1} by subdividing each edge (u, x_i) into two edges, i.e., replace the edge (u, x_i) by a path (u, v_i, x_i) of length 2. It was proved in [14] that for each integer $n \geq 1$, W_{2n+1} is a planar graph with circular chromatic number 4, and proved in [12] that for each $n \geq 2$, G_{2n+1} is a triangle free planar graph with circular chromatic number 3. (We note that this result follows easily from Theorem 11.) It was asked in [12] whether or not W_{2n+1} were the only minimal planar graphs with circular chromatic number 4, and whether or not G_{2n+1} were the only minimal triangle free planar graphs with circular chromatic number 3. Using the construction method

presented in this paper, one can easily construct infinitely many planar graphs with circular chromatic number 3 that do not contain triangles and do not contain any of the graphs G_{2n+1}, and infinitely many planar graphs with with circular chromatic number 4 that do not contain any of the odd wheels W_{2n+1}.

Fig. 3 are some examples constructed by this method. The graph in Fig. 3(a) is obtained from G_5 by replacing an edge with a 3-circular superedge. Hence it has circular chromatic number 3. Obviously, this graph is planar, triangle free and does not contain any of the graphs G_{2n+1} as subgraphs.

The graph in Fig. 3(b) is obtained from W_5 by replacing an edge with a 4-circular superedge. Hence it has circular chromatic number 4. Obviously, it is planar and does not contain any of the odd wheels W_{2n+1}.

The graph in Fig. 3(c) has circular chromatic number 4 by Theorem 11. This graph is also obtained from K_4 by replacing an edge with a special 4-circular superedge associated with 4 copies of K_4. Again this graph does not contain any of the odd wheels.
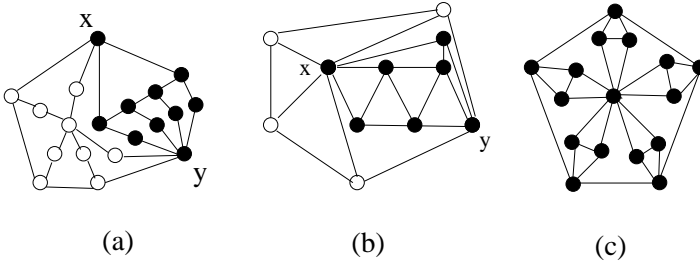

(a)   (b)   (c)

Fig. 3. Example of graphs constructed by using circular superedges

Remarks. 1. For any rational number $r = k/d \geq 2$, we can also define r-circular superedges. By using copies of G_k^d, and applying the operations of adding vertices and edges, identifying non-adjacent vertices, and replacing an edge with an r-circular superedge, we may obtain a class graphs of circular chromatic number at least r. Most of the discussion of this paper can be extended to the case of arbitrary rational numbers r.

2. The author learned from Kostochka that Mel'nikov has also constructed examples of planar graphs that answer in the negative the two questions of [12] mentioned above.

References

[1] H. L. ABBOTT and B. ZHOU: The star chromatic number of a graph, *J. Graph Theory,* **17** (1993), 349–360.

[2] J. A. BONDY and P. HELL: A note on the star chromatic number, *J. Graph Theory,* **14** (1990), 479–482.

[3] S. BRANDT: On the structure of dense triangle-free graphs, *Fachbereich Mathematik und Informatik, Series A Mathematik,* Preprint No. A 97–16.

[4] L. A. GODDYN, M. TARSI, and C. Q. ZHANG: On (k, d)-colorings and fractional nowhere-zero flows, *J. Graph Theory,* **28** (1998), 155–161.

[5] D. R. GUICHARD: Acyclic graph coloring and the complexity of the star chromatic number, *J. Graph Theory,* **17** (1993), 129–134.

[6] G. HAJÓS: Über eine Konstruktion nicht n-färbbarer Graphen, *Wiss. Zeitschrift Univ. Halle, Math.-Nat.,* **10** (1961), 113–114.

[7] P. HELL and X. ZHU: Circular chromatic number of series-parallel graphs, preprint, Department of Mathematics, National Sun Yat-sen University, 1998.

[8] T. R. JENSEN and B. TOFT: *Graph coloring problems*, Wiley 1995.

[9] A. JOHNSON, F. C. HOLROYD and S. STAHL: Multichromatic numbers, star chromatic numbers and Kneser graphs, *J. Graph Theory,* **26** (1997), 137–145.

[10] D. MOSER: The star-chromatic number of planar graphs, *J. Graph Theory,* **24** (1997), 33–43.

[11] J. PACH: Graphs whose every independent set has a common neighbour, *Discrete Math.,* **37** (1981), 217–228.

[12] E. STEFFEN and X. ZHU: Star chromatic numbers of graphs, *Combinatorica,* **16** (1996), 439–448.

[13] A. VINCE: Star chromatic number, *J. Graph Theory,* **12** (1988), 551–559.

[14] X. ZHU: Star chromatic numbers and products of graphs, *J. Graph Theory,* **16** (1992), 557–569.

[15] X. ZHU: Uniquely H-colorable graphs with large girth, *J. Graph Theory,* **23** (1996), 33–41.

[16] X. ZHU: Construction of uniquely H-colorable graphs, *J. Graph Theory,* to appear.

[17] X. ZHU: Planar graphs with circular chromatic numbers between 3 and 4, preprint, Department of Mathematics, National Sun Yat-sen University, 1997.

[18] X. ZHU: A simple proof of Moser's Theorem, *J. Graph Theory,* to appear.

[19] X. ZHU: Circular chromatic number and graph homomorphism, *Bulletin of Australian Mathematical Society,* to appear.

[20] X. ZHU: *Circular chromatic number and graph minor*, preprint, Department of Mathematics, National Sun Yat-sen University, 1997.

[21] X. ZHU: Circular chromatic number, a survey, preprint, Department of Mathematics, National Sun Yat-sen University, 1997.

Xuding Zhu

Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
zhu@math.nsysu.edu.tw